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1987 J. Phys. A: Math. Gen. 20 1743

(http://iopscience.iop.org/0305-4470/20/7/020)

J. Phys. A: Math. Gen. **20** (1987) 1743-1750. Printed in the UK

Objective interpretations of quantum mechanics and the possibility of a deterministic limit†

A Sudbery
Department of Mathematics, University of York, Heslington, York, YO1 5DD, UK

Received 23 June 1986, in final form 15 September 1986

Abstract. This paper examines a suggestion by Bell that a deterministic model might be furnished by the continuum limit of an objective stochastic model which he has proposed for quantum field theory on a lattice. The virtues of models of Bell's type for general quantum systems are discussed, and it is shown that for a system of a fixed number of particles such a model does yield the deterministic de Broglie-Bohm model in an appropriate limit. It is argued, however, that a deterministic limit is unlikely to emerge from a quantum field theory capable of describing particle production and decay. An interpretation of quantum field theory is proposed which has the advantages of Bell's interpretation but is closer to conventional quantum mechanics.

1. Introduction

Bell (1984) has proposed an interpretation of quantum field theory which makes it possible to formulate the theory without using the concept of measurement. The result is a unified theory in which the physical world is not divided into a 'system' and an 'observing apparatus' and its evolution is not divided into 'natural evolution', governed by the Schrödinger equation, and 'experiments', governed by the projection postulate. In this respect Bell's theory resembles the de Broglie-Bohm interpretation (Bohm 1952, Bell 1982) of non-relativistic quantum mechanics with a finite number of degrees. It differs from the de Broglie-Bohm theory in that it is not deterministic but probabilistic. However, Bell considers a field theory in which the spacetime continuum has been replaced by a lattice; the full theory is supposed to emerge in the limit of zero lattice spacing, and Bell expresses the hope that in this continuum limit his interpretation may become deterministic.

This paper is an attempt to examine the possibility of determinism emerging in the continuum limit. We first generalise Bell's proposal so as to be able to apply it to any quantum system (§ 2). Then, in § 3, we apply it to a system of a single particle and discuss the relation between a lattice model and the continuum limit. Two ways of defining the limit are considered; in both cases the probabilistic Bell model does seem to approach the deterministic de Broglie-Bohm model. In § 4 the significance of this result for models of quantum field theory is discussed; it is argued that the type of limit being considered is unlikely to be relevant to theories which describe processes in which the number of particles changes.

† A preprint version of this paper was circulated under the title 'Is there a deterministic model of quantum field theory?'.

0305-4470/87/071743+08$02.50

Many physicists, unlike Bell, find the determinism of the de Broglie–Bohm model an undesirable feature. Three-quarters of a century's experience with quantum reality has taught the undeniable lesson that events at this level are unpredictable. The same applies to the classical attributes of particles: there is now little desire to describe the world in terms of particles having a definite position, as in the de Broglie–Bohm model. On the other hand, it is probably a minority of physicists who would profess themselves content with the present state of quantum mechanics, with its dualistic and ill-defined projection postulate. Bell's interpretation, as generalised in § 2, seems to offer the possibility of a quantum mechanics which is realistic, being probabilistic and having only wavefunctions rather than particle positions, and yet precise and objective, with no projection postulate. Such an interpretation is outlined in § 5.

2. Bell's interpretation

Bell proposed to describe the state of the universe at time t by a complex comprising a quantum mechanical state vector $|\Phi(t)\rangle$ and a set of integers $n_{ix}(t)$, one for each basic fermion field ψ_i and each point $\boldsymbol{x}$ of space (regarded as a lattice). The integer n_{ix} is to be thought of as the value of $\tilde{\psi}_i(\boldsymbol{x})\psi_i(\boldsymbol{x})$, i.e. the number of particles of type i at the point $\boldsymbol{x}$ (here $\tilde{\psi}$ is the conjugate $\psi^\dagger\gamma_0$ of the Dirac spinor ψ). There is, as usual, a Hamiltonian operator H, and the state vector $|\Phi(t)\rangle$ evolves deterministically according to the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Phi(t)\rangle = H|\Phi(t)\rangle. \qquad (1)$$

The integers $n_{ix}(t)$, on the other hand, change stochastically; if n_{ix} has the value N at time t, the probability that it will have the value M at time $t+\mathrm{d}t$ is

$$T_{NM}\,\mathrm{d}t = \frac{P_{NM}}{D_N}\,\mathrm{d}t \qquad (2)$$

where

$$P_{NM} = \begin{cases} 2\,\mathrm{Re}\langle\Phi(t)|\Pi_{ix}(M)(i\hbar)^{-1}H\Pi_{ix}(N)|\Phi(t)\rangle & \text{if this is} \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$D_N = \langle\Phi(t)|\Pi_{ix}(N)|\Phi(t)\rangle$$

and $\Pi_{ix}(N)$ is the projection operator onto the eigenspace of $\tilde{\psi}_i(\boldsymbol{x})\psi_i(\boldsymbol{x})$ with eigenvalue N. (For further details see Bell (1984).)

In general, an interpretation like Bell's can be applied to any quantum system by supposing that there is a special set S of commuting observables which always take definite values. Let us call these observables the *visible properties* of the system. The state of the system at any time is then specified by a state vector $|\Phi(t)\rangle$ and a set of eigenvalues for the visible properties. Equivalently, the state of the system can be specified by the state vector $|\Phi(t)\rangle$ and one of its projections onto the simultaneous eigenspaces of the visible properties. Let us call this projection the *visible state* of the system and $|\Phi(t)\rangle$ the *pilot state*. We will take the pilot state to be normalised, so that the visible state is not normalised.

Let m and n denote sets of eigenvalues for the visible properties, and let Π_m be the projection operator onto the simultaneous eigenspace $\mathscr{S}_m$ with eigenvalues m. Then Bell's statement (2) of transition probabilities can be expressed as follows.

The probability that the visible state of the system is $|\phi_m\rangle$ at time t and $|\phi_n\rangle$ at time $t+\mathrm{d}t$, if $m \neq n$, is $P_{mn}\,\mathrm{d}t$ where

$$P_{mn} = 2\,\mathrm{Re}[(\mathrm{i}\hbar)^{-1}\langle\phi_n|H|\phi_m\rangle] \tag{3}$$

provided that this is $\geqslant 0$ and that $|\phi_m\rangle = \Pi_m|\Phi(t)\rangle$, $|\phi_n\rangle = \Pi_n|\Phi(t+\mathrm{d}t)\rangle$; otherwise $P_{mn} = 0$.

Note that P_{mn} is obtained from the time derivative of $|\langle\phi_n|H|\phi_m\rangle|^2$. It can be shown (Bell 1984, Sudbery 1986b) that the above statement implies that if the pilot state vector $|\Phi(t)\rangle$ is expanded as

$$|\Phi(t)\rangle = \sum_n |\phi_n(t)\rangle \qquad \text{with } |\phi_n(t)\rangle \in \mathcal{S}_n$$

then the visible state at time t is $|\phi_n(t)\rangle$ with probability $\langle\phi_n(t)|\phi_n(t)\rangle$, provided that these probabilities hold at $t=0$. It follows that the density matrix for the visible state is

$$\begin{aligned}\rho(t) &= \sum_n |\phi_n(t)\rangle\langle\phi_n(t)| \\ &= \sum \Pi_n|\Phi(t)\rangle\langle\Phi(t)|\Pi_n.\end{aligned} \tag{4}$$

The necessity for an interpretation of this kind has been argued elsewhere (Sudbery 1984, 1986a) in connection with the problem of continuous observation. It was also argued that this does not involve any unorthodox addition to the usual theory, since such an interpretation is normally, though tacitly, adopted in the course of calculations of decay rates. It also occurs more explicitly in the derivation of master equations (Joos 1984) where it is assumed that the density matrix is of the form (4). We will return to this point in § 4.

3. The continuum limit

In order to see how it is possible for a deterministic model to be the continuum limit of a stochastic lattice model like that of § 2, we will study a simple continuous quantum system, a single particle moving in a one-dimensional potential, which is known to have a deterministic model (the de Broglie–Bohm model). We will consider two ways of approximating this system by a model like Bell's; both yield something like the de Broglie–Bohm model in an appropriate limit.

In the one-dimensional de Broglie–Bohm model the state of the particle is specified by a wavefunction $\psi(x, t)$ and a real number $X(t)$ (the position of the particle on the line). These change deterministically according to the equations

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = H\psi = \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\psi\right) \tag{5}$$

$$\frac{\mathrm{d}X}{\mathrm{d}t} = \frac{j(X, t)}{\rho(X, t)} \tag{6}$$

where

$$j = \hbar/m\,\mathrm{Im}[\bar{\psi}\,\mathrm{d}\psi/\mathrm{d}x] \qquad \rho = |\psi|^2 \tag{7}$$

(the bar denotes the complex conjugate). It follows that the probability that X lies between x and $x+\mathrm{d}x$ at time t is $|\psi(x, t)|^2\,\mathrm{d}x$, provided that this is true at $t=0$.

The first lattice model of this system consists of replacing the continuous line by a discrete series of points x_k with spacing ε and considering only the values of the wavefunction on this one-dimensional lattice. Labelling the points of the lattice by the integer k, we have a state vector consisting of a doubly infinite sequence $\phi(k)$. To approximate the usual integral $\int \overline{\phi_1(x)}\phi_2(x)\,\mathrm{d}x$, we take the scalar product to be given by

$$\langle\phi_1|\phi_2\rangle = \varepsilon \sum_k \overline{\phi_1(k)}\phi_2(k). \tag{8}$$

Defining the second difference operator Δ_2 by

$$\Delta_2\psi(k) = \varepsilon^{-2}[\psi(k+1) - 2\psi(k) + \psi(k-1)]$$

(not the square of a first difference operator), we take the Hamiltonian to be

$$H = -\frac{\hbar^2}{2m}\Delta_2 + V \tag{9}$$

where V is the potential: $V\psi(k) = V(x_k)\psi(k)$. The stochastic law of evolution, modelled on (2), is that if the particle is at x_k at time t the probability that it is at the different point x_l at time $t+\mathrm{d}t$ is

$$\frac{P_{kl}}{D_k}\mathrm{d}t = \frac{2\,\mathrm{Re}(\mathrm{i}\hbar)^{-1}\langle\Phi(t)|\Pi_l H \Pi_k|\Phi(t)\rangle\,\mathrm{d}t}{\langle\Phi(t)|\Pi_k|\Phi(t)\rangle} \qquad \text{if this is} \geqslant 0 \tag{10}$$

where

$$(\Pi_k\phi)(m) = \phi(m)\delta_{km}.$$

From (8) and (9) we find, if V is real,

$$P_{kl} = \mathrm{Re}\left(\frac{\mathrm{i}\hbar}{m\varepsilon}\overline{\phi(l)}\phi(k)(\delta_{k,l+1} - 2\delta_{kl} + \delta_{k,l-1})\right) \tag{11}$$

$$D_k = \varepsilon|\phi(k)|^2 = \text{probability that the particle is at } x_k.$$

Thus if the particle is at x_k it only makes first-order transitions to the neighbouring points x_{k+1} and x_{k-1}; the probability of transition to x_{k+1} in time $\mathrm{d}t$ is

$$\begin{aligned}
&\frac{\mathrm{d}t}{\varepsilon|\phi(k)|^2}\mathrm{Re}\left(\frac{\mathrm{i}\hbar}{m\varepsilon}\overline{\phi(k+1)}\phi(k)\right)\\
&\quad = \frac{\mathrm{d}t}{\varepsilon|\phi(k)|^2}\mathrm{Im}\left(\frac{\hbar}{m\varepsilon}\overline{\phi(k)}[\phi(k+1) - \phi(k)]\right) \qquad \text{if this is} \geqslant 0\\
&\quad = (j_+(k)/\rho(k))(\mathrm{d}t/\varepsilon) \qquad \text{if } j_+(k) \geqslant 0
\end{aligned} \tag{12}$$

where

$$\rho(k) = |\phi(k)|^2$$

and

$$j_+(k) = (\hbar/m\varepsilon)\,\mathrm{Im}[\overline{\phi(k)}\Delta\phi(k)]$$

$$\Delta\phi(k) = \phi(k+1) - \phi(k).$$

The probability of transition to x_{k-1} is

$$(j_-(k)/\rho(k))(\mathrm{d}t/\varepsilon) \qquad \text{if } j_-(k) \leqslant 0 \tag{13}$$

where

$$j_-(k) = (\hbar/m\varepsilon)\,\mathrm{Im}[\overline{\phi(k)}\Delta\phi(k-1)].$$

As $\varepsilon \to 0$, both $j_+(k)$ and $j_-(k)$ tend to $j(x_k)$, where j is the usual current given by (7). The probabilities (12) and (13) are the appropriate ones for a particle which moves forwards (if $j > 0$) or backwards (if $j < 0$) in steps of ε occurring at random but so as to maintain an average velocity of j/ρ, as in the de Broglie–Bohm model.

Clearly this model is trying to become the de Broglie–Bohm model as $\varepsilon \to 0$. Nevertheless, it could be argued that for all values of ε, no matter how small, it remains a stochastic model *mimicking* a deterministic model rather than actually *becoming* deterministic.

It might seem more sensible, instead of pursuing the lattice spacing to zero in a literal interpretation of the 'continuum limit' as a limit, to go straight to a continuum model with the same structure as the lattice model, probabilities being replaced by probability densities. By taking the state vector $|\Phi(t)\rangle$ to be a wavefunction of the continuous variable x, and taking the projection operators to project onto δ-functions, one might expect to be able to write down a probability that the particle is in an interval $[a, b]$ at time $t + \mathrm{d}t$ if it is at a definite position x at time t. However, this approach runs into difficulty in interpreting the positivity condition in (10); this has to be applied to a distribution $\delta''(x-y)$. To get a meaningful model it is necessary to relax the condition that the particle is at a given position at time t and suppose only that it is in a given interval, as in the following.

In this second Bell-type model of the system, the pilot state space is the usual space of wavefunctions ϕ of a continuous variable x, and the visible state is a wavefunction that vanishes outside one of a set of intervals I_k into which the line is divided by partition points x_k. Thus the model supposes that the particle is always in one of the intervals I_k and that it makes transitions between them with probabilities given by (3). In order to avoid problems with products of distributions we make the intervals I_k overlap by a small amount 2δ; thus $I_k = [x_k - \delta, x_{k+1} + \delta]$ and the projection operators Π_k are given by

$$\Pi_k\phi(x) = \theta(x - x_k + \delta)\theta(x_{k+1} + \delta - x)\phi(x). \tag{14}$$

According to (3) the probability that there is a transition from I_k to I_l in time $\mathrm{d}t$ is

$$P_{kl}\,\mathrm{d}t = \hbar^{-1}\,\mathrm{Im}\langle\phi|\Pi_l H \Pi_k|\phi\rangle\,\mathrm{d}t \qquad \text{if this is } \geqslant 0.$$

This vanishes unless I_k and I_l are adjacent; if $l = k-1$, then

$$\begin{aligned}\langle\Phi|\Pi_{k-1}H\Pi_k|\Phi\rangle &= \int_{x_{k-1}-\delta}^{x_k+\delta} \overline{\phi(x)}\left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V\right)[\phi(x)\theta(x - x_k + \delta)]\,\mathrm{d}x \\ &= -\frac{\hbar^2}{2m}\left[\bar\phi\frac{\mathrm{d}\phi}{\mathrm{d}x} - \frac{\mathrm{d}\bar\phi}{\mathrm{d}x}\phi\right]_{x=x_k-\varepsilon} + \int_{x_k-\delta}^{x_k+\delta}\overline{\phi(x)}H\phi(x)\,\mathrm{d}x \\ &\to -\mathrm{i}(\hbar^2/m)\,\mathrm{Im}[\overline{\phi(x_k)}\phi'(x_k)] \qquad \text{as } \delta \to 0.\end{aligned}$$

Thus

$$\lim_{\delta\to 0} P_{k-1,k} = -j(x_k) \qquad \text{if } j(x_k) \leqslant 0. \tag{15}$$

Similarly,

$$\lim_{\delta\to 0} P_{k+1,k} = j(x_{k+1}) \qquad \text{if } j(x_{k+1}) \geq 0. \tag{16}$$

These are just the probabilities that would be obtained from the de Broglie–Bohm model, for in that model the probability of transition from I_k to I_{k-1} is non-zero only if $v_k = j(x_k)/\rho(x_k)$ is negative, and then it is the probability of there being a particle within a distance $-v_k\,dt$ to the right of x_k, namely $-v_k\,dt\,\rho(x_k) = -j(x_k)\,dt$. Similarly, (16) gives the probability per unit time of a particle moving from I_k to I_{k+1} across x_{k+1}.

Thus it is possible to obtain the deterministic de Broglie–Bohm model as a limit of a Bell-type model for a single particle in one dimension. Note, however, that the limit had to be defined in a very *ad hoc* fashion. There is no *a priori* justification for making the intervals I_k overlap; we could just as well approach the limit of abutting intervals by leaving gaps of length 2δ between the intervals and letting $\delta\to 0$. But then the transition probabilities would be zero for any finite δ, and so they would also be zero in the limit.

Clearly these arguments can be extended to a system with any number of degrees of freedom, and in particular to a set of n particles moving in space.

4. The continuum limit in quantum field theory

The non-relativistic single-particle system considered in the previous section can be regarded as a sector of a field theory system described by a single non-relativistic fermion field $\psi(x)$ (in the Schrödinger picture) with Hamiltonian

$$H = \int \left(-\frac{\hbar^2}{2m}\psi^\dagger \frac{d^2\psi}{dx^2} + V(x)\psi^\dagger\psi \right) dx. \tag{17}$$

In this theory particle number is conserved, so the n-particle subspaces with different values of n are decoupled; the usual first-quantised theory is the one-particle sector.

In the lattice model the continuous set of field operators $\psi(x)$ becomes a discrete set $\psi_k = \psi(x_k)$. On the one-particle subspace the particle number operators $\psi_k^\dagger\psi_k$ are just the projection operators Π_k of equation (10). Their eigenvalues, of course, are 0 and 1. These are the visible properties of the system. As the lattice spacing tends to 0 the Bell stochastic process in the one-particle sector approaches the de Broglie–Bohm deterministic process, at least to the extent that this is true of the first-quantised theory of § 3. The same is true of the n-particle sector for each n. Thus the field-theoretic model as a whole can be held to have a deterministic limit. The fact that its visible properties have discrete spectra does not make it intrinsically less deterministic than a theory with continuous visible properties.

Nevertheless, these indications of a deterministic limit for the system with Hamiltonian (17), for which the number of particles is constant, do not give good grounds for a belief that such a limit exists for other forms of quantum field theory. When the particle number is fixed, the deterministic continuum theory is one which is readily visualised, and it is intuitively plausible that a lattice structure for space should require the continuum theory to be modified so that deterministic continuous motion is replaced by stochastic jumping. On the other hand, if the Hamiltonian induces processes in which particle number is not conserved (including the particle–antiparticle pair production processes of relativistic quantum field theory), then it is hard to imagine the form of a deterministic theory which describes such processes, and it is hard to see the relevance of whether space is a lattice or a continuum to whether particle production and decay are stochastic or deterministic. These are not conclusive arguments, but there are no arguments at all in the other direction, and to me it seems most plausible that, unless there is some radical departure from the formalism of quantum field theory, particle production and decay will have to be described as stochastic processes.

5. Must we believe in particles with definite positions?

The advantages of an objective interpretation like Bell's over the various forms of the Copenhagen interpretation are the following.

(i) There is no need to specify what is meant by a 'measurement'. The conventional interpretations do not even exist until it is specified what physical processes are to count as measurements; yet any particular definition is bound to be highly implausible, and in fact I have never seen a detailed proposal for a precise definition of 'measurement' (but see Bussey 1984, Maxwell 1982). Thus Bell's interpretation has the advantage that it exists.

(ii) Conventional interpretations give a thoroughly implausible account of quantum jumps. The only allowance they make for transitions from one state to another is that they occur as a result of experimental intervention with the system in the form of a measurement. Nevertheless, *pace* Schrödinger (1952), quantum jumps occur spontaneously and will be seen to do so by an observer who watches the system continuously without intervening (Joos 1984, Sudbery 1984, 1986a). Objective interpretations recognise this situation and (if the idea of a deterministic limit is abandoned) describe it in accordance with what seems to be the fact of experience that quantum transitions are unpredictable.

By now it seems to be also a fact of experience (though maybe it is only a prejudice instilled by our education) that a quantum particle does not have a definite position but can only be described by a wavefunction. This makes the de Broglie–Bohm theory hard to accept; although it has exactly the same observable consequences as conventional quantum mechanics, it seems implausible because it introduces extra objective properties without offering any way of exhibiting their separate existence. Is it not possible to keep the advantages of an objective interpretation without having to believe in particle positions?

One way of doing this would be to keep as objective properties the total numbers of particles of each kind, without specifying their positions any more definitely than by giving a wavefunction. This can be done in the framework of the generalised Bell interpretation of § 2, taking the visible properties to be the various particle numbers, i.e. the integrated fermion densities† $\int \tilde{\psi}_i(\boldsymbol{x})\psi_i(\boldsymbol{x})\, \mathrm{d}^3\boldsymbol{x}$ (this only emphasises the lack of Lorentz covariance which is already a feature of Bell's interpretation, and which might be remediable by making objective properties relative to the frame of reference in a suitable way). Then quantum events consist of the creation and annihilation of fermions, in line with the meaning of 'event' in particle physics; only these events are not localised, but are described by a wavefunction. Alernatively, one could construct

† It might seem that by taking the visible properties to be the total particle numbers (in the whole universe), we lose the ability to represent the fact that there are more fermions in some (macroscopically defined) places than others. However, the visible *state* of the universe includes the wavefunctions of the fermions, and information about the macroscopic localisation of the fermions is contained in their wavefunctions. This interpretation, which contains more objective properties of the world than conventional quantum mechanics, cannot be less capable of representing facts about the world.

a theory in which the basic events were the emission and absorption of photons or other bosons by taking the total photon number as visible property.

In such a theory a quantum jump, for example the decay of an excited state of an atom, is an objective event occurring at a definite time. It should be realised, however, that this time is not necessarily what is determined experimentally as the time of decay. If an excited state $|\psi\rangle$ of an atom evolves in time t, according to the Schrödinger equation, to

$$c(t)|\psi\rangle+|\psi'(t)\rangle \tag{18}$$

where $|\psi'(t)\rangle$ is a state of the atom plus a photon, then $|\psi'(t)\rangle$ can be written as

$$|\psi'(t)\rangle=\int_0^t c(t')|\psi_0\rangle|\gamma(t-t')\rangle\,\mathrm{d}t' \tag{19}$$

where $|\psi_0\rangle$ is the ground state of the atom and $|\gamma(\tau)\rangle$ is the state of the photon in which it has travelled a distance $c\tau$ from the atom (see Sudbery (1986b) § 3.5). It will be necessary to do an experiment on the photon (which means, in this interpretation, to elicit a further quantum transition) to decide between the various possibilities presented in the superposition (19). Thus the system may make a quantum jump at time t to a state in which it appears to have made a quantum jump at an earlier time t'. This is true whether or not particle positions are taken to be visible properties, as they are in the Bell and de Broglie–Bohm models.

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